Cryptography

7 – Loose ends

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Today

- Elliptic curves
- Key management
- Proofs
- Homomorphic encryption
- And more...

Recall: Generalized DLP

Let $(\mathcal{G}, \cdot\,)$ be a finite abelian group.

Given $g \in \mathcal{G}$ and x such that

$$x = g^{\xi} = \underbrace{g \cdot g \,\cdots\, g}_{\xi} \quad \text{in} \;\; \mathcal{G},$$

find $\xi \underset{\nu}{\equiv} \log_g(x)$, with $\nu = \mathrm{ord}_{\mathcal{G}}(g)$, the smallest $\nu > 0$ for which $g^{\nu} = 1$.

Best known DL algorithm: $\mathcal{O}(\nu^{\frac{1}{2}})$ for a generic group $\mathcal{G}$. (Much smaller for $\mathcal{G} = (\mathbb{Z}/n\mathbb{Z})^{\times}$.)

Elliptic curves

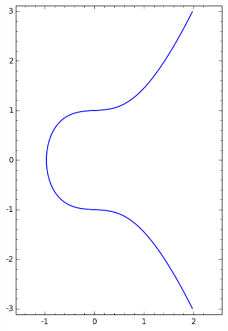
Definition

An *elliptic curve* is a plane curve defined by an equation of the form

$$\mathcal{E}: \ y^2 = x^3 + ax + b.$$

Example

$a = \frac{1}{10}, \ b = 1$



Addition on an elliptic curve

Given $P, Q \in \mathcal{E}$, the line through P and Q intersects $\mathcal{E}$ at a third point, say $R = (x, y)$.

Definition

$$P + Q := (x, -y)$$

Fun fact: This makes $\mathcal{E} \cup \{O\}$ into an abelian group!

(The *point at infinity* $O = (0, \infty)$ being the neutral element)

DLP on an elliptic curve

Given $G \in \mathcal{E}$ of (additive) order n and $P \in \mathcal{E}$ such that

$$P = mG = \underbrace{G + \cdots + G}_{m} \quad \text{in} \;\; \mathcal{E},$$

find $m \underset{n}{\equiv} \log_G(P)$.

(Easy to solve over the real or complex numbers)

Elliptic curves over finite fields

Instead: consider solutions modulo a fixed prime p

$$y^2 \underset{p}{\equiv} x^3 + ax + b$$

$\rightsquigarrow \mathcal{E}(\mathbb{F}_p)$ elliptic curve over the finite field $\mathbb{F}_p$

(a finite abelian group!)

Basic computations are easy…

```
p = 32806352226718822643429
a = 5740347588375554626864
b = 20798093206103976495852

E = EllipticCurve(GF(p),[a,b])

P = E([29155336995917130553754, 8373057744944244479010])
Q = E([3415221595160200314960, 11073266156995522792160])

2*P + 3*Q
```

Evaluate   Language: Sage

Share

```
(9956939019642126506349 : 26680698275736540367982 : 1)
```

Help | Powered by SageMath

…but the DLP is hard!

Size of $\mathcal{E}$

Theorem (Hasse bound)

$$\#\mathcal{E}(\mathbb{F}_p) = 1 + p + \mathcal{O}(\sqrt{p})$$

hence $\#\mathcal{E}(\mathbb{F}_p) \approx p$.

We use elliptic curves with points G of large order $n \approx p$.

ECDH

- Alice and Bob agree on "safe" parameters $\mathcal{E}$ and G.
- Alice chooses a, computes $A = aG$ in $\mathcal{E}$.
- Bob chooooses b, computes $B = bG$ in $\mathcal{E}$.
- Shared secret is

$$K := (ab)G = aB = bA.$$

ECElGamal

Keys:

- d private decryption key
- $E = dG$ public encryption key

Alice wants to send a message $M \in \mathcal{E}$ to Bob.

ECElGamal

Encryption:

- Alice chooses random s, computes $S = sG$
- Computes shared secret $K = sE$
- Computes encrypted $C = M + K$
- Sends the pair (S, C)

Decryption:

Upon reception of a pair (S, C), Bob

- Computes shared secret $K = dS$
- Recovers $M = C - K$

Parameter generation

To get ℓ bits of security:

- choose a 2ℓ-bit prime p
- an elliptic curve $\mathcal{E}$ over $\mathbb{F}_p$
- and a point G on $\mathcal{E}$ of (almost) prime order n that generates (most of) $\mathcal{E}(\mathbb{F}_p)$.

Much harder to manufacture than *e.g.* for RSA – but can be reused.

Recommended curves

In the US, NIST proposed in 2005 a list of 5 elliptic curves of size

$$192,\ 224,\ 256,\ 384 \text{ and } 521 \text{ bits}$$

(as well as 5 curves over binary fields $\mathbf{F}_{2^k}$)

$$\cdots$$

Dual_EC_DRBG controversy

Alternative: Brainpool curves

Also: recent concern about Suite B *cf.* rise of quantum computing!?

Post-quantum cryptography

Ongoing NIST standardization process for quantum-resistant primitives.

Round 2: 17 public-key encryption primitives, 9 digital signature primitives.

Broadly fall into 4 categories:

- lattice-based
- code-based
- hash-based
- multivariate polynomial-based

Stay tuned!

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Key management

Consider a pool of n users, each of which could want to communicate confidentially with any other.

$\implies \binom{n}{2} = \frac{n(n-1)}{2}$ interactions to secure.

With a single secret key for every potential interaction:

every user needs to securely obtain and store $n-1$ *secret* keys!

Purely asymmetric solution

Use public-key encryption for everything.

Every user needs access to any of the $n - 1$ other *public* keys

But: asymmetric ciphers are much slower than symmetric ones.

$\implies$ *hybrid* systems are usually favored (but: full-fledged PKI needed)

Example: TLS/SSL

TLS 1.3 specification

- X.509 certificates are used to authenticate the parties
- A **master secret** is set up
- Bulk of communication encrypted with a symmetric cipher
- MACs are included for data integrity

Various combinations of cipers and MACs (**cipher suites**) are supported (providing varying levels of security).

Cipher suite: example

- RSA-PSS signature for server authentication
- ECDH for key agreement
- Sessions keys are derived from the master secret
- AES-CBC used for encryption
- SHA256-HMAC for message authentication

Agreed upon during initial *handshake*.

Comments

- Provides **forward secrecy** if fresh DH parameters are used every time (recommended!)
- These parameters are signed, preventing man-in-the-middle attacks
- Session keys need to be refreshed after a while
- Often subject to *downgrade attacks*

Kerberos

Purely symmetric key management solution using a trusted **key server** S

Alice wants to communicate securely with Bob.

- Both set up secret keys k_A and k_B with the server.
- Alice asks the server for a secret key k_{AB} to be used with Bob.

Needham-Schroeder algorithm (1978)

- The server replies to Alice with

$$E\big(k_A,\, k_{AB} \,\|\, E(k_B, k_{AB})\big).$$

- Alice decrypts this message and sends to Bob

$$E(k_B, k_{AB}).$$

Alice and Bob now have k_{AB} and can start communicating securely.

Comments

- Nonces need to be included to prevent *replay attacks*
- Provides mutual authentication as well as confidentiality
- Man-in-the-middle attacks are not possible
- Server does not need to remember keys
- But: single point of failure

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How to trust others' computations?

In various cryptographic protocols, Bob might worry that Alice is not doing things properly

(read: cheats! – or makes mistakes)

and ask her for *proofs* of good conduct.

Bob: *challenger*

Alice: *prover*

Infamous example: proof of work

To make sure that Alice has access to suitable computing resources:

on input m, asks her to find a string k for which the binary representation of

$$H(m \,\|\, k) \text{ starts with } n \text{ zeros.}$$

Partial collision problem: her best approach is to brute-force k

will take 2^n trials on average

(this is what Bitcoin cryptominers do... with an ecological impact of epic proportions)

Example: coin flip

Alice and Bob play a game.

Heads: A gives €100 to B, tails: B gives €100 to A.

Alice is responsible for tossing the coin.

Alice: "Tails!"

Bob: "Prove it!"

Secure coin flip

- Alice chooses a random large integer n
- Sends its SHA256 hash to Bob (*commitment*)
- Bob selects $b \in \{0, 1\}$, sends it to Alice
- Alice returns $(n \% 2) \oplus b$ (result of coin toss)

  and n (proof of randomness)

Alice cannot manipulate the result unless she knows n and n' of different parity with the same hash!

Zero-knowledge proofs

Sometimes Alice wants to convince Bob of a certain statement, *without revealing anything else than the fact that this statement is true.*

Example

Alice: "I know ξ such that $g^{\xi} \underset{p}{\equiv} x$"

Bob: "Prove it!"

Zero-knowledge proof

Idea: Bob should present Alice with requests that she can only answer correctly if she does indeed know ξ – and that Bob can check are answered correctly.

- Alice chooses a random number $\rho \in]\!]0, q[\![$ and sends $c \underset{p}{\equiv} g^{\rho}$ to Bob.
- Bob randomly requests Alice to either disclose

$$\rho \qquad \text{or} \qquad \rho + \xi \qquad \mod q.$$

Correctness

If Bob receives exponent ρ' from Alice, he can check the agreement with *commitment* c by computing

$$g^{\rho'} \qquad \text{or} \qquad g^{\rho'} \cdot x^{-1} \qquad \bmod p.$$

Alice can easily fake a correct answer (without knowing ξ) to any of those questions *but not both*. She would have to guess correctly which question Bob will ask before to commit an adequate value of c.

If Alice answers correctly n requests in a row, Bob can trust that the probability that she knows ξ is $\geq 1 - \frac{1}{2^n}$.

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Malleability, revisited

We mainly considered malleability a bad thing.

But it can actually be useful!

Example

Alice wants to compute the product of two ℓ-bit integers m_1 and m_2. She could

- Encrypt them using plain-RSA with a 2ℓ-bit modulus
- Send the ciphertexts to Bob and ask *him* to multiply them
- Decrypt the resulting ciphertext.

Homomorphic encryption

Certain ciphers preserve addition *or* multiplication.

Definition

A **fully homomorphic** cipher is one that preserves both addition and multiplication.

So what?

A cryptographer's dream

1978

Suppose we have a fully homomorphic cipher

$$E : \mathcal{M} = (\mathbf{F}_2, \oplus, \odot) \longrightarrow \mathcal{C}.$$

Then, since

$$\begin{cases} x \texttt{ and } y = x \odot y \\ x \texttt{ or } y = x \oplus y \oplus (x \odot y) \\ \texttt{not } x = 1 \oplus x \end{cases}$$

we can build a processor that works with encrypted bits!

Fast-forward to 2009

Theorem (C. Gentry, Standford Ph.D. thesis)

Fully homomorphic ciphers exist.

Gentry's original construction used lattice-based cryptography but a more elementary one was later found.

In both approaches, one starts with a *somewhat homomorphic cipher*.

Somewhat homomorphic encryption

Secret key: a large odd integer k

Encryption: to encrypt $b \in \{0, 1\}$, choose random q and m with $2m \in [\![0, k-1[\![$ and set

$$c = qk + 2m + b.$$

Decryption: $b = (c \,\%\, k) \,\%\, 2$

Bootstrapping

These encrypted bits can support a limited number of operations while still decrypting correctly.

After that: need to refresh encryption.

How to do that in the blind processor?

Decrypt through the encryption!

Refreshing encryption

- Alice sends $c_1 = E(k_1, b)$ to Bob
- Bob computes $c_{12} = E(k_2, c_1)$
- Then computes $c_2 = D(k_1, c_{12})$ *through the encryption* in order to get

$$c_2 = E(k_2, b).$$

(For this to work, an asymmetric version of the cipher needs to be used)

A somewhat homomorphic cipher only needs to support its own decryption circuit *plus one operation*.

So is this used everywhere in the cloud?

Not yet. . . still an area of active research & development.

Current implementations are still somewhat impractical (slow / large keys)

One could in principle run arbitrary encrypted code on arbitrary encrypted data on a remote processor and get the encrypted result back!

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And more...

- split secrets
- secure multipartite computation
- identity and attribute-based encryption
- digital currencies (blockchain)
- differential privacy
- quantum cryptography

New Crypto Wars episode coming soon to a computer near you . . .